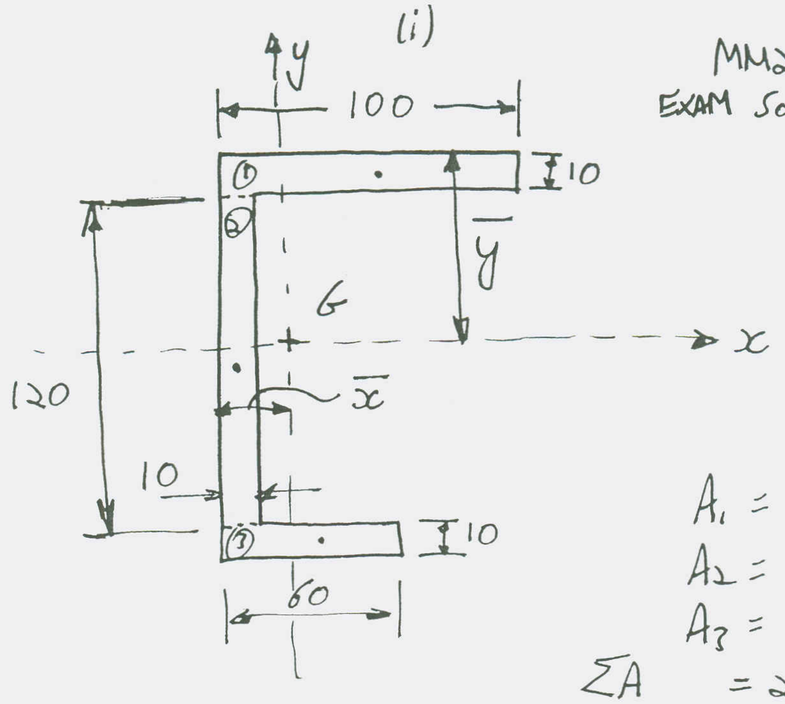
MM2SM2 08-09
EXAM SOLUTIONS

Q1

(i)



$A_1 = 1000$
$A_2 = 1200$
$A_3 = 600$
$\Sigma A = \underline{2800}\ mm^2$

Position of Centroid

$$\bar{x} = (A_1 . 50 + A_2 . 5 + A_3 . 30)/\Sigma A$$
$$= (50{,}000 + 6{,}000 + 18{,}000)/2800$$
$$= \underline{26.43\ mm} \text{ from left edge}$$

$$\bar{y} = (A_1 . 5 + A_2 . 70 + A_3 . 135)/\Sigma A$$
$$= (5000 + 84{,}000 + 81{,}000)/2800$$
$$= \underline{60.71\ mm} \text{ from top edge}$$

[7 MARKS]

2nd Moments of Area

$$I_x = \left(\frac{100.10^3}{12} + 1000.(55.71)^2\right) + \left(\frac{10.120^3}{12} + 1200.(9.29)^2\right) + \left(\frac{60.10^3}{12} + 600.(74.29)^2\right)$$
$$= 3.112.10^6 + 1.544.10^6 + 3.316.10^6$$
$$= \underline{\underline{7.972.10^6}}$$

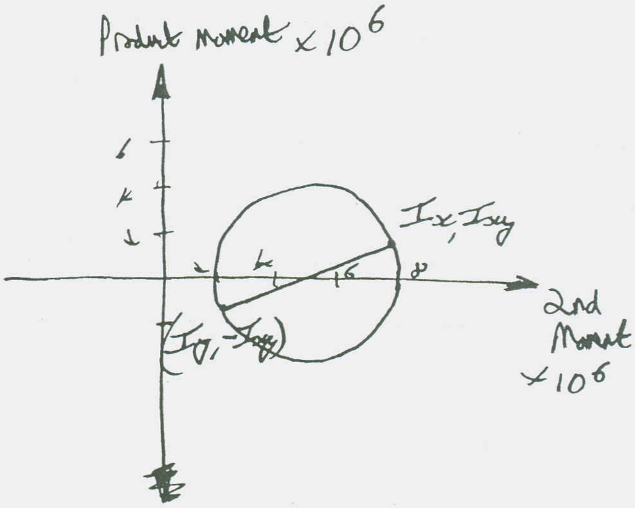
$$I_y = \left(\frac{10.100^3}{12} + 1000(23.57)^2\right) + \left(\frac{120.10^3}{12} + 1200.(21.43)^2\right) + \left(\frac{10.60^3}{12} + 600.3.57^2\right)$$
$$= 1.389.10^6 + 0.561.10^6 + 0.188.10^6$$
$$= \underline{\underline{2.138.10^6}}$$

Q1 (cont)

$$I_{xy} = 1000.(+23.57).(+55.71) + 1200(-21.43)(-9.29) + 600(+3.57)(-74.29)$$

$$= 1.313.10^6 + 0.239.10^6 - 0.159.10^6$$

$$= \underline{\underline{1.393.10^6}}$$



$$C = \frac{I_x + I_y}{2} = \underline{5.055.10^6}$$

$$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$= \sqrt{(2.417)^2 + (1.393)^2}$$

$$= \underline{3.233}$$

$$\therefore I_p = C + R = \underline{\underline{8.288.10^6 \, mm^4}}$$

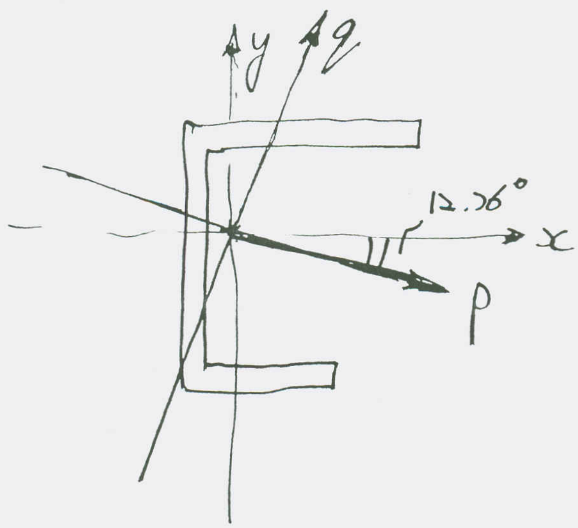
$$I_q = C - R = \underline{\underline{1.828.10^6 \, mm^4}}$$

[12 MARKS]

$$\sin 2\theta = \frac{I_{xy}}{R} = \frac{1.393}{3.233} = 0.431 \quad \therefore 2\theta = 25.52°$$

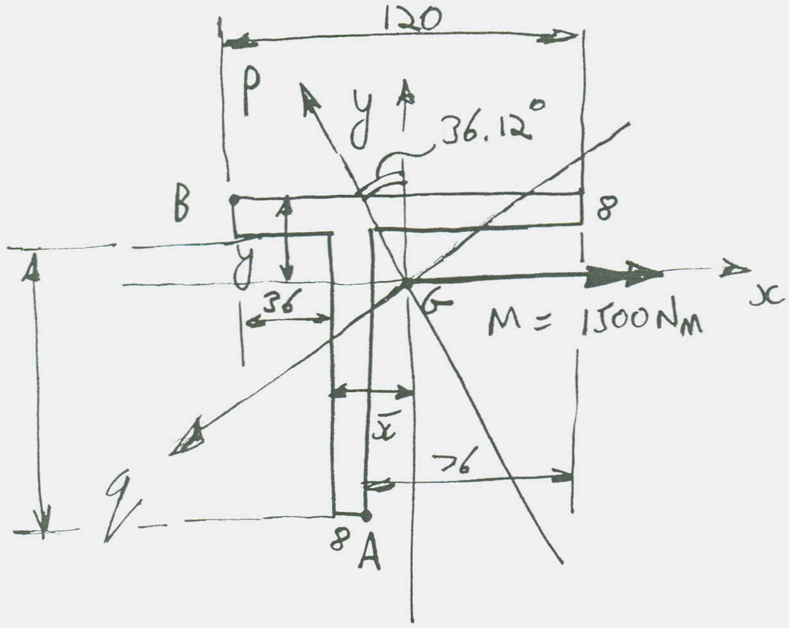
$$\therefore \theta = \underline{\underline{12.76°}}$$

Sketch



[6 MARKS]

Q2 (Soln)



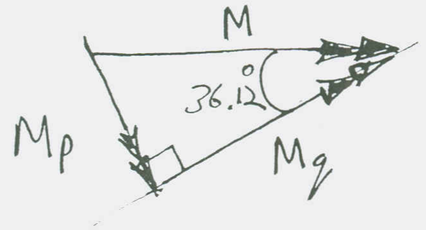
$\bar{x} = 16\ \text{mm}$

$\bar{y} = 21.6\ \text{mm}$

$I_p = 1.557 \cdot 10^{-6}\ \text{m}^4$

$I_q = 0.843 \cdot 10^{-6}\ \text{m}^4$

Bending Moment Components



$M_p = -1500 \sin 36.12^\circ = -884.2\ \text{Nm}$

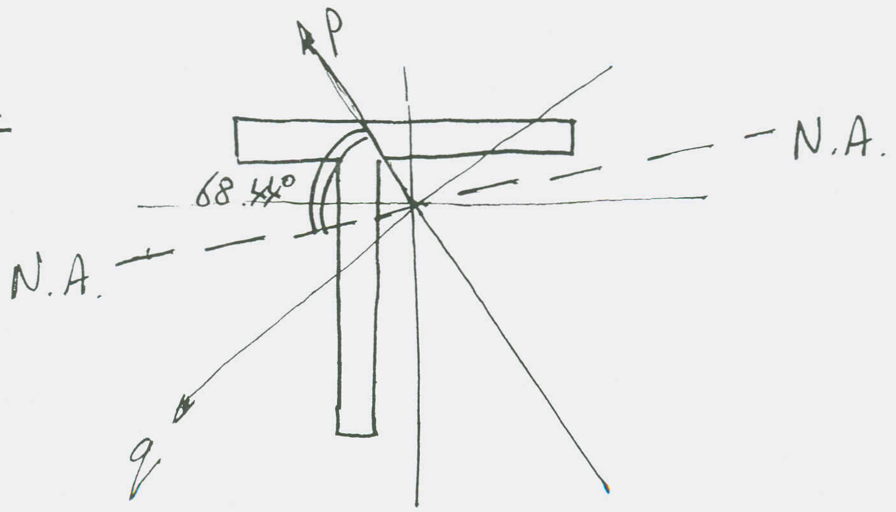
$M_q = -1500 \cos 36.12 = -1211.7$

Angle of Neutral Axis

$$\alpha = \tan^{-1}\left[\frac{q}{p}\right] = \tan^{-1}\left[\frac{M_q I_p}{M_p I_q}\right] = \tan^{-1}\left[\frac{-1211.7}{-884.2} \cdot \frac{1.557}{0.843}\right]$$

$$= \underline{68.44^\circ}$$

Sketch



[7 MARKS]

Q2 (Soln cont)

σ_b at point A

Co-ordinates of A (mm)

x	y	p	q
-8	-66.4	-48.94	45.57

using
$p = x\cos\theta + y\sin\theta$
$q = -x\sin\theta + y\cos\theta$

$\sin\theta = \sin(126.12) = \underline{0.808}$
$\cos\theta = \cos(126.12) = \underline{-0.589}$

$$\therefore \sigma_{b_A} = \frac{M_p\, q}{I_p} - \frac{M_q\, p}{I_q}$$

$$= \frac{(-884.2)(45.57 \cdot 10^{-3})}{1.557 \cdot 10^{-6}} - \frac{(-1211.7)\cdot(-48.94 \cdot 10^{-3})}{0.843 \cdot 10^{-6}}$$

$$= -25.88 - 70.34 \quad . 10^{6}$$

$$= \underline{\underline{-96.22\ MPa}} \quad \text{ie. compressive}$$

[9 MARKS]

Maximum Tensile Stress at point B
– Furthest from N.A.

Co-ordinates of B (mm)

x	y	p	q
-52	21.6	48.08	29.29

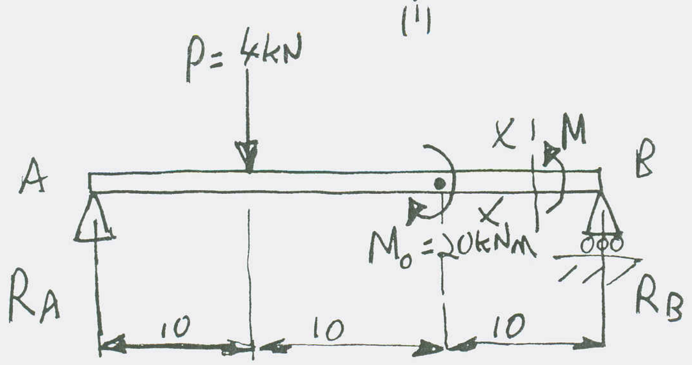
$$\sigma_{b_B} = \frac{-884.2 \cdot (29.29 \cdot 10^{-3})}{1.577 \cdot 10^{-6}} - \frac{(-1211.7)\cdot(48.08 \cdot 10^{-3})}{0.843 \cdot 10^{-6}}$$

$$= -16.42 + 69.11 \quad . 10^{6}$$

$$= \underline{\underline{52.69\ MPa}} \quad \text{tensile}$$

[9 MARKS]

Q3 Soln



Reaction Forces

Moments about B

$R_A . 30 + 20.10^3 - 4.10^3 . 20 = 0$

$30 R_A + 20.10^3 - 80.10^3 = 0$

$R_A = 60.10^3/30 = \underline{\underline{2000N}}$

3 marks

Moments about section X–X
at distance x from A

$EI =$

$$M - M_0 - R_A x + P\langle x-10\rangle = 0$$

4 marks

not singularity function

$$\therefore EI\frac{d^2y}{dx^2} = -M = -R_A x - M_0\langle x-20\rangle^0 + P\langle x-10\rangle$$

$$EI\frac{dy}{dx} = -\frac{R_A x^2}{2} - M_0\langle x-20\rangle^1 + \frac{P\langle x-10\rangle^2}{2} + A$$

$$EIy = -\frac{R_A x^3}{6} - \frac{M_0\langle x-20\rangle^2}{2} + \frac{P\langle x-10\rangle^3}{6} + Ax + B$$

8 marks

Boundary Conditions.

when $x = 0$ $y = 0$ $\therefore$ $B = 0$

when $x = 30$ $y = 0$ $\therefore$ $0 = -\frac{R_A 30^3}{6} - \frac{M_0 10^2}{2} + \frac{P 20^3}{6} + A.30$

$$\therefore 30A = \frac{30^3 . 2000}{6} + \frac{20.10^3 . 100}{2} - \frac{4000.20^3}{6}$$

$$30A = 9.10^6 + 1.10^6 - 5.33.10^6$$

$$30A = 4.67.10^6$$

$$\therefore A = \underline{1.557 . 10^5}$$

4 marks.

When $x = 10$

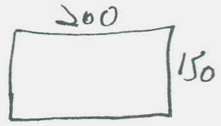
$$EIy = -\frac{2000.10^3}{6} - \cancel{\frac{20.10^3(-10)^2}{2}}^{\,0} + \cancel{\frac{4000(0)}{6}}^{\,0} + 1.557.10^5 . 10$$

Q3 (contd)

$$EIy = -0.333 \cdot 10^{6} + 1.557 \cdot 10^{6}$$

$$EIy = \underline{1.224 \cdot 10^{6}}$$

For a rectangular section $I = \dfrac{bd^3}{12} = \dfrac{0.2 \cdot 0.15^3}{12}$



$$= 5.625 \cdot 10^{-5}$$

$$\therefore \; EI = 210 \cdot 10^{9} \cdot 5.625 \cdot 10^{-5} = \underline{\underline{1.181 \cdot 10^{7} \; Nm^2}}$$

3 Marks

$$\therefore \; y = \frac{1.224 \cdot 10^{6}}{1.181 \cdot 10^{7}} = 0.104$$

$$= \underline{\underline{104 \text{ mm}}}$$

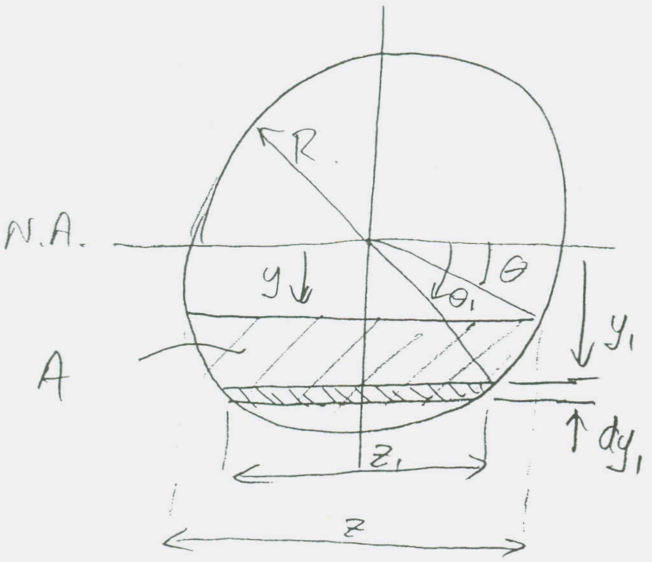
3 marks

[25 MARKS]

SM2 08-09

Q4

(a)



The shear stress at y is

$$\tau = \frac{S}{I_z} A\bar{y}$$

$$= \frac{S}{I_z} \int_A y \, dA$$

Change variable to simplify integration:

$$y_1 = R\sin\theta_1$$
$$dy_1 = R\cos\theta_1 \, d\theta_1$$
$$z_1 = 2R\cos\theta_1$$
$$z = 2R\cos\theta$$

$$I = \frac{\pi D^4}{64} = \frac{\pi R^4}{4}$$

$$\tau = \frac{4S}{\pi R^4 \cdot 2R\cos\theta} \int_\theta^{\frac{\pi}{2}} \underbrace{R\sin\theta_1}_{y_1} \cdot \underbrace{2R\cos\theta_1}_{z_1} \cdot \underbrace{R\cos\theta_1 \, d\theta_1}_{dy_1}$$

$$= \frac{4S}{\pi R^2 \cos\theta} \int_\theta^{\frac{\pi}{2}} \cos^2\theta_1 \cdot \sin\theta_1 \, d\theta_1$$

Let $u = -\cos\theta_1 \Rightarrow \frac{du}{d\theta_1} = \sin\theta_1 \Rightarrow du = \sin\theta_1 \, d\theta_1$

$$\Rightarrow \int \cos^2\theta_1 \sin\theta_1 \, d\theta_1 = \int u^2 du = \frac{u^3}{3} = \frac{-\cos^3\theta_1}{3}$$

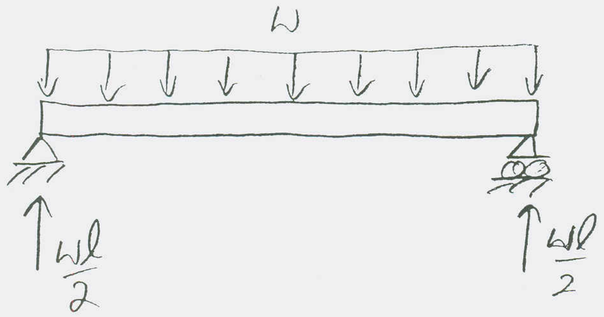
$$\Rightarrow \tau = \frac{4S}{\pi R^2 \cos\theta} \left[-\frac{\cos^3\theta_1}{3} \right]_\theta^{\frac{\pi}{2}} = \frac{4S}{3\pi R^2} \cos^2\theta$$

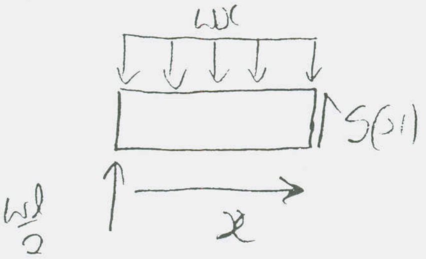
$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{y}{R}\right)^2$$

$$\Rightarrow \quad \tau = \frac{4S}{3\pi R^2}\left[1 - \left(\frac{y}{R}\right)^2\right]$$

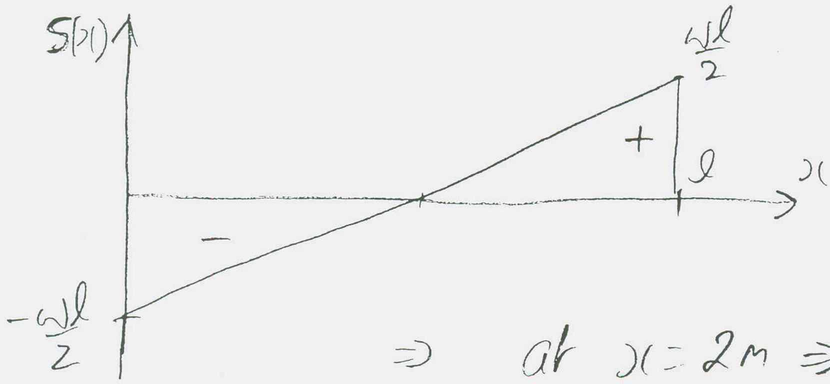
[12 MARKS]

(b)





$$S(x) = wx - \frac{wl}{2}$$



$$\Rightarrow \quad \text{at } x = 2m \Rightarrow S = 2w - 3w$$
$$= -w$$
$$= -3000 \text{ N}$$

$$\tau_{max} = \frac{4S}{3\pi R^2} = 100 \text{MPa}$$

$$\Rightarrow \quad R^2 = \frac{12000}{300\pi} = 12.73 \text{mm}^2$$

$$\Rightarrow R = 3.57 \text{mm}$$

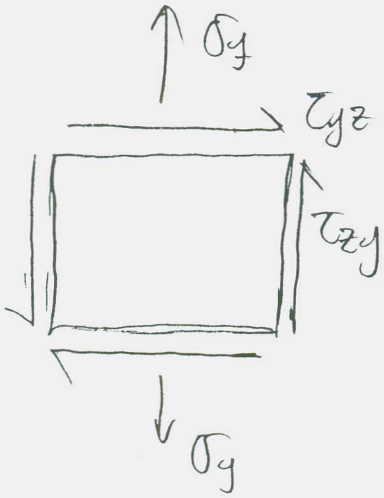
[13 MARKS]

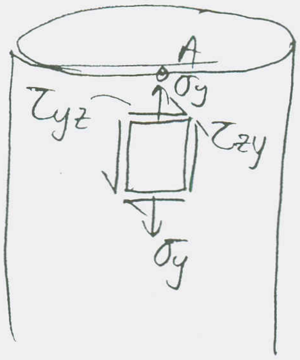
Q5

(a)

$$\sigma_y = \frac{My}{I} + \frac{P}{A} \quad (1)$$

$$\tau_{yz} = \frac{TR}{J} = \frac{TD_o}{2J} \quad (1)$$





$$\sigma_y = \frac{2TD_o}{2I} + \frac{3T}{D_o \frac{\pi}{4}(D_o^2 - D_i^2)} \quad (1)$$

$$I = \frac{\pi}{64}(D_o^4 - D_i^4)$$

$$J = \frac{\pi}{32}(D_o^4 - D_i^4)$$

$$\tau_{yz} = \frac{TD_o}{2J} \quad (1)$$

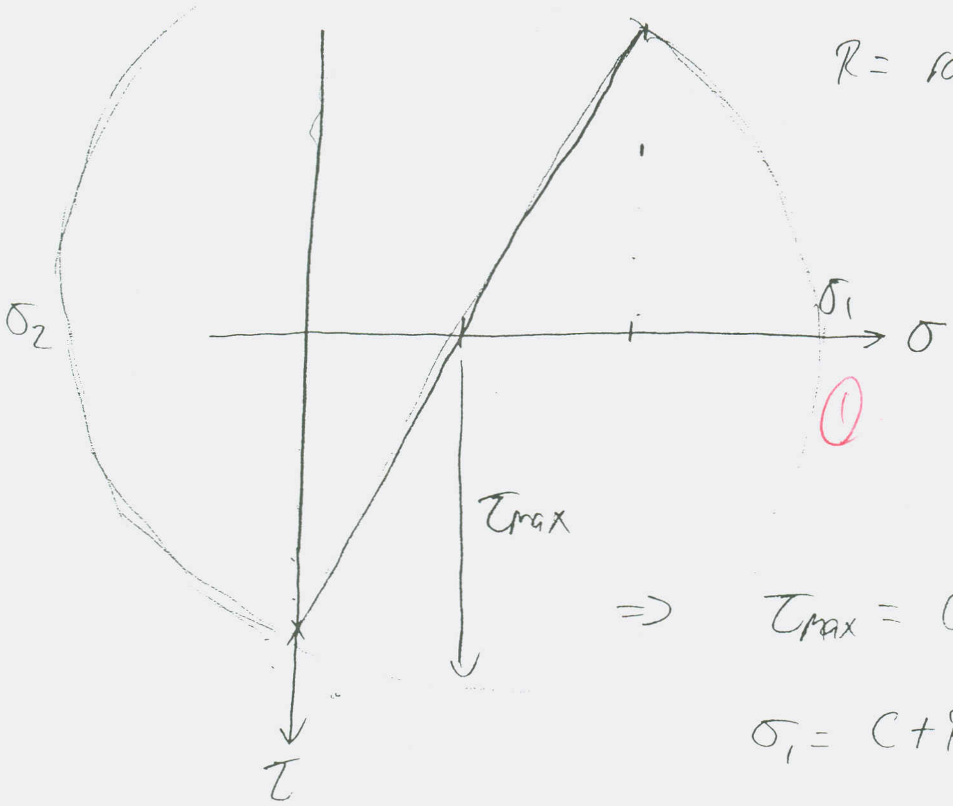
$$\left.\begin{array}{l} D_o = 30 \text{ mm} \\ D_i = 20 \text{ mm} \end{array}\right\} \Rightarrow \begin{array}{l} A = 392.7 \text{ mm}^2 \\ I = 31906.8 \text{ mm}^4 \\ J = 63813.6 \text{ mm}^4 \end{array} \quad (1)$$

$$\Rightarrow \sigma_y = \frac{30T}{31906.8} + \frac{3T}{30(392.7)} = 0.00094T + 0.000225T$$

$$= 0.00119T \quad (1)$$

$$\tau_{yz} = 0.000235T \quad (1)$$

Mohr's circle: C = centre: $\frac{\sigma_y}{2} = 0.000595T$ 1



R = radius $= \sqrt{\left(\frac{\sigma_z - \sigma_y}{2}\right)^2 + \tau_{yz}^2}$ 1

$= \sqrt{(0.000595T)^2 + (0.000235T.}$

$= 0.00064T$ 1

$\Rightarrow \quad \tau_{max} = 0.00064T$ 1 64T Pa

$\sigma_1 = C + R = 0.00123T$ 1 123T Pa

$\sigma_2 = C - R = -0.000045T$ 1 −45T Pa

[17 MARKS]

(b)

Max. shear stress yield criterion:

yield occurs when $\tau_{max} = \frac{\sigma_y}{2}$

$\Rightarrow \quad 0.00064T = \frac{1000}{2}$ MPa

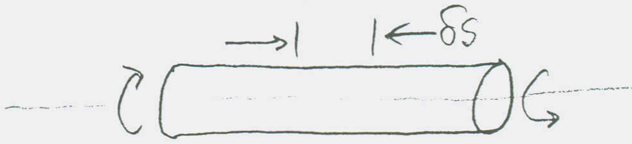
$\Rightarrow \quad T = 781250$ Nmm

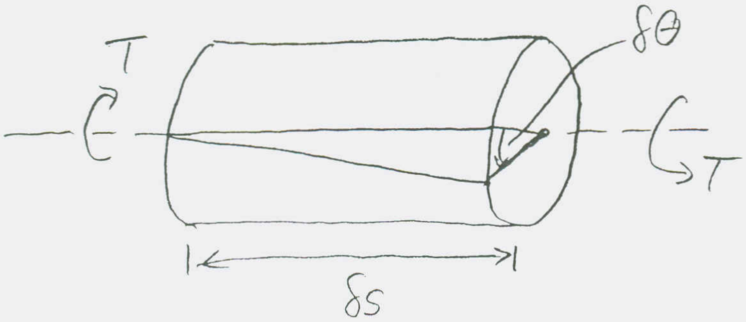
$= 781.25$ Nm

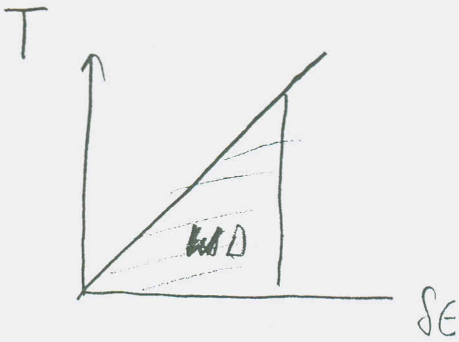
[8 MARKS]

Q6

(a)







$$W.D. = \frac{1}{2} T \delta\theta$$

$$\Rightarrow \quad \delta U = \frac{1}{2} T \delta\theta$$

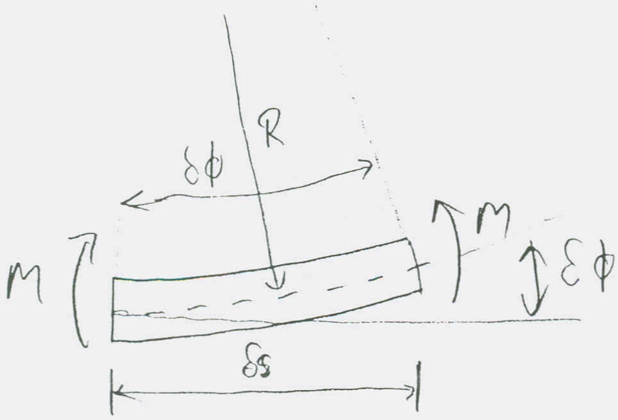
But $\quad \delta\theta = \frac{T}{GJ} \delta s$

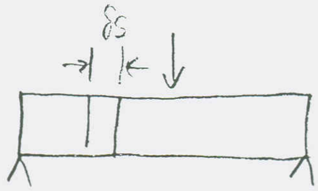
$$\Rightarrow \quad \delta U = \frac{T^2}{2GJ} \delta s$$

$$\Rightarrow \quad U = \int_0^L \frac{T^2}{2GJ} \delta s.$$

[7 MARKS]

(b)





$$\delta U = \frac{1}{2} M \delta\phi$$

But $\quad R\delta\phi = \delta s \quad$ and $\quad M = \frac{EI}{R}$

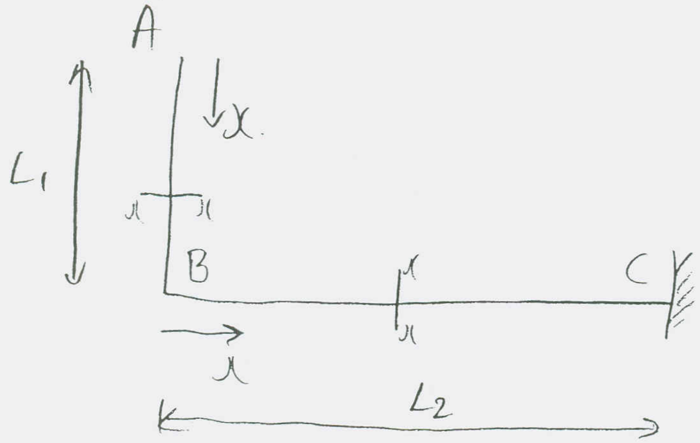
$$\Rightarrow \quad \delta\phi = \frac{\delta s}{R} = \frac{M}{EI} \delta s$$

$$\Rightarrow \quad \delta U = \frac{M^2}{2EI}\delta s$$

$$\Rightarrow \quad U = \int_0^L \frac{M^2}{2EI}\delta s$$

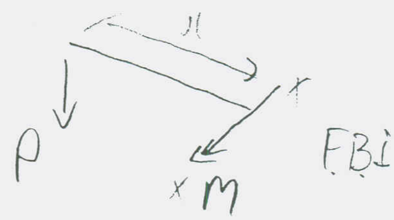
[7 MARKS]

(C)



For section AB

$\circlearrowleft_{x-x}$ $M + Px = 0$ torque $T = 0$



$\therefore \quad M = -Px$

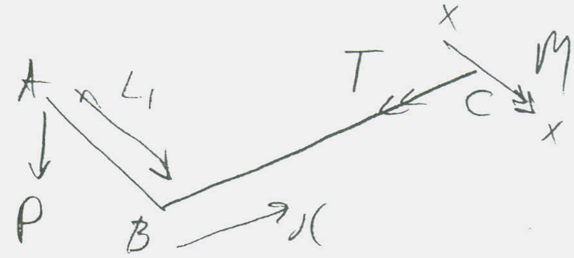
$$\therefore \quad U_{AB} = \int \frac{M^2}{2EI} ds$$

$$= \int_0^{L_1} \frac{P^2x^2}{2EI_{AB}} ds$$

$$= \frac{P^2L_1^3}{6EI_{AB}}$$

8 ④

For $\underline{BC}$



$$M + Px = 0 \therefore M = -Px$$

$$T + PL_1 = 0 \therefore T = -PL_1$$

$$U_{BC} = \int_0^{L_2} \frac{M^2}{2EI}\,dx + \int_0^{L_2} \frac{T^2}{2GJ}\,dx$$

$$= \int_0^{L_2} \frac{P^2 x^2}{2EI_{BC}}\,dx + \int_0^{L_2} \frac{P^2 L_1^2}{2GJ_{BC}}\,dx$$

$$= \frac{P^2 L_2^3}{6EI_{BC}} + \frac{P^2 L_1^2 L_2}{2GJ_{BC}} \qquad 4$$

$$\text{Total } U = P^2 \left\{ \frac{L_1^3}{6EI_{AB}} + \frac{L_2^3}{6EI_{BC}} + \frac{L_1^2 L_2}{2GJ_{BC}} \right\}$$

$$u_V = \frac{\partial U}{\partial P} = 2P \left\{ \frac{L_1^3}{6EI_{AB}} + \frac{L_2^3}{6EI_{BC}} + \frac{L_1^2 L_2}{2GJ_{BC}} \right\} \qquad 1$$

$$I_{AB} = \frac{\pi}{64}(20^4 - 10^4) = 7363.1 \text{ mm}^4 \qquad L_1 = 200\text{mm}$$

$$I_{BC} = \frac{\pi}{64}(30^4 - 20^4) = 31906.8 \text{ mm}^4 \qquad L_2 = 400\text{mm}$$

$$J_{BC} = 63813.6 \text{ mm}^4 \qquad 1$$

$$\Rightarrow u_V = 2(1000)\left\{ \frac{(200)^3}{6(200\times10^3)(7363)} + \frac{(400)^3}{6(200\times10^3)(31906.8)} + \frac{(200)^2(400)}{2(70\times10^3)(6381} \right.$$

$$\Rightarrow \quad u_v = 2(1000)\left\{0.000905 + 0.00167 + 0.00179\right\}$$

$$= 8.73\,\text{mm.} \quad 1$$

[11 MARKS]